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EFFICIENT ANALYTIC COMPUTATION OF HIGHER-ORDER QCD AMPLITUDES*

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ABSTRACT

We review techniques simplifying the analytic calculation of one-loop QCD amplitudes with many external legs, for use in next-to-leading-order corrections to multi-jet processes. Particularly useful are the constraints imposed by perturbative unitarity, collinear singularities and a supersymmetry-inspired organization of helicity amplitudes. Certain sequences of one-loop helicity amplitudes with an arbitrary number of external gluons have been obtained using these constraints.

1. Total Quantum-number Management

The calculation of one-loop QCD amplitudes with many external quarks and gluons is a bottleneck that must be navigated in order to obtain next-to-leading-order (NLO) corrections to multijet processes, for precision comparison with collider experiments. The full correction has a real (bremsstrahlung) part as well as a virtual part. Efficient techniques for computing the tree amplitudes entering the real part have been available for several years[2]; however, significant numerical work is required to combine these parts into a finite answer. In this talk we ignore the numerical subtleties,[a] and focus on techniques for computing analytically the one-loop amplitudes entering the virtual part.

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[a]Such subtleties have recently been discussed for the energy-energy correlation in e^+e^- annihilation[1].

In principle it is straightforward to compute one-loop amplitudes by drawing all Feynman diagrams and evaluating them using standard reduction techniques for the loop integrals. In practice this method becomes extremely inefficient and cumbersome as the number of external legs grows, because there are:
1. **too many diagrams** — many diagrams are related by gauge invariance, and
2. **too many terms in each diagram** — nonabelian gauge boson self-interactions are complicated.
Consequently, intermediate expressions tend to be vastly more complicated than the final results, when the latter are represented in an appropriate way.

A useful organizational framework, that helps tame the size of intermediate expressions, is Total Quantum-number Management (TQM), which suggests to:
• Keep track of all possible information about external particles — namely, *helicity* and *color* information.
• Keep track of quantum *phases* by computing the transition amplitude rather than the cross-section.
• Use the helicity/color information to decompose the amplitude into simpler, gauge-invariant pieces, called *sub-amplitudes* or *partial amplitudes*.
• Square amplitudes to get probabilities, and sum over helicities and colors to obtain unpolarized cross-sections, only at the very *end* of the calculation.
Carrying out the last step explicitly would generate a large analytic expression; however, at this stage one would typically make the transition to numerical evaluation, in order to combine the virtual and real corrections. The use of TQM is hardly new, particularly in tree-level applications[2] — but it is especially useful at loop level.

As an example, consider the one-loop amplitude for n external gluons, all taken to be outgoing. We generalize the $SU(3)$ color group to $SU(N_c)$, and label the gluons $i = 1, 2, \ldots, n$ by their adjoint color indices $a_i = 1, 2, \ldots, N_c^2-1$, and helicities $\lambda_i = \pm$. The helicity decomposition uses gluon circular polarization vectors expressed in terms of massless Weyl spinors[3]. The color decomposition[4] is performed in terms of traces of $SU(N_c)$ generators T^a in the fundamental representation, with $\mathrm{Tr}(T^aT^b) = \delta^{ab}$,

$$\mathcal{A}_n^{1-\mathrm{loop}}\left(\{k_i,\lambda_i,a_i\}\right) = g^n \Bigg[\sum_{\sigma\in S_n/Z_n} N_c \, \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right) \; A_{n;1}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))$$
$$+ \sum_{c=2}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}}\right) \, \mathrm{Tr}\left(T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(n)}}\right) \;\; A_{n;c}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})) \Bigg] \,,$$

where $A_{n;c}$ are the partial amplitudes, g is the gauge coupling, S_n is the set of all permutations of n objects, while Z_n and $S_{n;c}$ are the subsets of S_n that leave the corresponding single and double trace structures invariant.

The $A_{n;1}$ are more basic, and are called *primitive amplitudes*, because:
a. They only receive contributions from diagrams with a particular cyclic ordering of the gluons around the loop, which greatly simplifies their analytic structure.
b. The remaining $A_{n;c>1}$ can be generated[4,5] as sums of permutations of the $A_{n;1}$.[b]

[b] For amplitudes with external quarks as well as gluons, the primitive amplitudes are not a subset of

Even the $A_{n;1}$ are not all independent, due to parity and cyclic invariance. For example, for $n=5$ only four are independent, $A_{5;1}(1^+,2^+,3^+,4^+,5^+)$, $A_{5;1}(1^-,2^+,3^+,4^+,5^+)$, $A_{5;1}(1^-,2^-,3^+,4^+,5^+)$, and $A_{5;1}(1^-,2^+,3^-,4^+,5^+)$. The first two of these are not required at NLO because the corresponding tree helicity amplitudes vanish, and are very simple for the same reason. For n_f quark flavors, they are given by[7]

$$A_{5;1}(1^+,2^+,3^+,4^+,5^+) = \frac{iC}{48\pi^2}\frac{\langle 1\,2\rangle\,[1\,2]\,\langle 2\,3\rangle\,[2\,3] + \langle 4\,5\rangle\,[4\,5]\,\langle 5\,1\rangle\,[5\,1] + \langle 2\,3\rangle\,\langle 4\,5\rangle\,[2\,5]\,[3\,4]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,,$$

$$A_{5;1}(1^-,2^+,3^+,4^+,5^+) = \frac{iC}{48\pi^2}\frac{1}{\langle 3\,4\rangle^2}\left[-\frac{[2\,5]^3}{[1\,2]\,[5\,1]} + \frac{\langle 1\,4\rangle^3\,[4\,5]\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2} - \frac{\langle 1\,3\rangle^3\,[3\,2]\,\langle 4\,2\rangle}{\langle 1\,5\rangle\,\langle 5\,4\rangle\,\langle 3\,2\rangle^2}\right],$$

where $C = 1 - \frac{n_f}{N_c}$ and $\langle j\,l\rangle$, $[j\,l]$ are spinor inner products[3,2]. Analytic expressions for the other two primitive amplitudes are more complex but still “fit on a page”[7]. In contrast, the color- and helicity-summed virtual correction to the cross-section, built from permutation sums of the two primitive amplitudes, would fill hundreds of pages.

2. Analytic Properties (and Supersymmetry)

There are at least five different ways to compute the partial/primitive amplitudes:

1. Traditional Feynman diagrams (in the helicity, color-ordered basis).
2. Rules derived from superstring theory[8].
3. Rules inspired by superstring theory[9].
4. Recursive construction[10] (see also the talk by Mahlon[11] in these proceedings).
5. Exploitation of their analytic properties (and supersymmetry).

Here we just discuss route 5, which can be the most efficient route to the answer.

The analytic behavior of loop amplitudes includes both cuts and poles. Since primitive amplitudes are “color-ordered” (property a), they have cuts and poles only in channels formed by the sum of *cyclically adjacent* momenta, $(k_i + \cdots + k_{i+r-1})^2$. In a (massless) supersymmetric theory, because of the improved ultraviolet behavior, the cuts alone are enough to reconstruct the full one-loop amplitude[13]. The cuts are computable in closed form in many cases. For example, the infinite sequence of maximal helicity-violating amplitudes in $N=4$ super-Yang-Mills theory are given[5] (in $4-2\epsilon$ dimensions, through $\mathcal{O}(\epsilon^0)$) by a sum of known scalar box integral functions $F_{n:r;i}$ (j and k are the only gluons with negative helicity):

$$A_{n;1}^{N=4}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\,\frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,V_n\,,$$

where

$$\begin{aligned} V_{2m+1} &= \sum_{r=2}^{m-1}\sum_{i=1}^{n} F^{\mathrm{2m}\,e}_{n:r;i} + \sum_{i=1}^{n} F^{\mathrm{1m}}_{n:i}\,, \\ V_{2m} &= \sum_{r=2}^{m-2}\sum_{i=1}^{n} F^{\mathrm{2m}\,e}_{n:r;i} + \sum_{i=1}^{n} F^{\mathrm{1m}}_{n:i} + \sum_{i=1}^{n/2} F^{\mathrm{2m}\,e}_{n:m-1;i}\,. \end{aligned}$$

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the partial amplitudes; new color-ordered objects have to be defined[6].

Supersymmetric results can be used to trade QCD calculations with internal gluons for somewhat easier calculations where scalars replace the gluons. For an amplitude with all external gluons, we rewrite the internal gluon loop g (and fermion loop f) as a supersymmetric contribution plus a complex scalar loop s,

$$\begin{aligned} g &= (g+4f+3s) \;-\; 4(f+s) \;+\; s \;=\; [N=4] \;-\; 4\,[N=1] \;+\; s, \\ f &= (f+s) \;-\; s \;=\; [N=1] \;-\; s, \end{aligned}$$

where $[N=1]$ represents the contribution of an $N=1$ chiral supermultiplet. The scalar contribution cannot be reconstructed directly from its cuts because of an additive "polynomial" ambiguity. It seems possible to fix this ambiguity by inspecting the factorization (pole) behavior of the amplitude, namely the limits where two (or more) adjacent momenta become collinear. The general form of these limits for one-loop amplitudes has recently been proven[15]. For the special case of identical gluon helicities, $(1^+,\ldots,n^+)$, the limits were successfully used to construct an ansatz[14] which was subsequently confirmed by recursive techniques[10]. If one can show that polynomial ambiguities for arbitrary helicity configurations can be determined uniquely and efficiently from factorization limits, then one would have a general technique for constructing one-loop QCD amplitudes without ever evaluating genuine loop diagrams.

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